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A Lattice Hierarchy with a Free Function and Its Reductions to the Ablowitz-Ladik and Volterra Hierarchies

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Abstract By embedding a free function into a compatible zero curvature equation, we propose a lattice hierarchy with the free function which still admits zero curvature representation. It is interesting that the hierarchy can reduce the Ablowitz-Ladik hierarchy, the Volterra hierarchy and a new hierarchy by properly choosing the embedded function. Moreover, the new hierarchy is integrable in Liouville's sense and possess multi-Hamiltonian structure.

Keywords Spectral problem · Nonlinear lattice hierarchy · Ablowitz-Ladik hierarchy · Volterra hierarchy · Multi-Hamiltonian structure

1 Introduction

In recent years, there has been much interest in the study of nonlinear lattice systems such as the Ablowitz-Ladik lattice, the Toda lattice, relativistic Toda lattice, the Volterra lattice, discrete KdV equation and the Suris lattice etc. [1–23]. It is well-known that most of these discrete integrable systems are related to the following discrete spectral problem

$$E\psi_n = U_n(u_n, \lambda)\psi_n, \tag{1.1}$$

and its auxiliary problem

$$\psi_{n,t} = V_n(u_n, \lambda)\psi_n, \tag{1.2}$$

where $u_n = (u_{1,n}, u_{2,n}, \dots, u_{p,n})^T$ are potentials and λ is a spectral parameter. The integrability condition between (1.1) and (1.2) leads to the discrete zero curvature equation

$$U_{n,t} - (EV_n)U_n + U_nV_n = 0. \tag{1.3}$$

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It is most interesting to find proper matrixes U_n and V_n, such that (1.3) associates with a lattice hierarchy of physical significance. It is also an important topic to explore the relations between different lattice hierarchies.

In this paper, we are interested in investigating a discrete soliton hierarchy and its reductions. In general, if the term $-(EV_n)U_n + U_nV_n$ is compatible with the term $U_{n,t}$, the zero curvature equation (1.3) leads to a discrete hierarchy. At this rate, the extra modification to the matrix V_n is not needed any more. However, we note that the matrix V_n which satisfies (1.3) is not unique, it is possible to embed a free function into the matrix V_n such that the derived new lattice hierarchy still admits the zero curvature representation (1.3) and some interesting reductions.

The spectral problem we consider in this paper takes

$$E\psi_n = U_n\psi_n = \begin{pmatrix} \lambda & r_n \\ s_n & \frac{q_n}{\lambda} \end{pmatrix} \psi_n, \tag{1.4}$$

where r_n, s_n, q_n are potentials and E is a shift operator defined by $Ef(n,t) = f(n+1,t) \equiv f_{n+1}$. Recently, the spectral problem (1.4) was used to study the integrable discretization of the derivative Schrödinger equation and the N-soliton solution of a discrete system by Tsuchida, Ito and Kakuhata [24, 25]. Zhang and Tu ever investigated the symmetries and conservation of a hierarchy related to the following spectral problem [5]

$$\tilde{\psi}_{n+1} = \tilde{U}_n\tilde{\psi}_n = \begin{pmatrix} \lambda + \tilde{s}_n & \tilde{q}_n \\ \tilde{r}_n & 1 \end{pmatrix} \tilde{\psi}_n. \tag{1.5}$$

Here we should point out that the spectral (1.4) and (1.5) are similar in form, but they are not gauge equivalent each other. That is why we are interested in the spectral problem (1.4) in this paper.

In the following Sect. 2, with the help of spectral problem (1.4), we derive a lattice hierarchy with a free function which includes the Ablowitz-Ladik hierarchy, Volterra hierarchy and a new hierarchy as special cases. In Sect. 3, we further show that the new hierarchy is integrable in Liouville's sense and possesses multi-Hamiltonian structure.

2 The Lattice Hierarchy Associated with the Spectral Problem (1.4) and Its Reductions

Let us introduce some basic notations and definitions to be used in this paper. We define the difference operator Δ by

$$\Delta = (E+1)(E-1)^{-1}, \quad n \in Z.$$

It is easy to show that the operators E and Δ possess the following properties

$$E^* = E^{-1}, \qquad \Delta^* = -\Delta, \qquad (E+1)^{-1} = \sum_{k=0}^{\infty}(-1)^k E^k, \qquad (1-E)^{-1} = \sum_{k=0}^{\infty} E^k,$$

where E^* and Δ^* denote adjoint operators of E and Δ respectively.

For a scalar function f_n, its variational derivative with respective to u_n is defined by

$$\frac{\delta f_n}{\delta u_n} = \sum_{k\in Z} E^{-k}\frac{\partial f_n}{\partial u_{n+k}}.$$

Let $f_n = (f_{1,n}, \ldots, f_{p,n})^T$ and $g_n = (g_{1,n}, \ldots, g_{p,n})^T$ be two vector functions, then their inner product is defined by

$$(f_n, g_n) = \sum_{n\in Z}\sum_{i=1}^{p} f_{i,n} g_{i,n}.$$

For any two scalar functions f_n and g_n, the expression

$$\{f_n, g_n\} = \left(J\frac{\delta f_n}{\delta u}, \frac{\delta g_n}{\delta u}\right) = \sum_{n\in Z}\sum_{i=1}^{p}\left(J\frac{\delta f_n}{\delta u}\right)_i \left(\frac{\delta g_n}{\delta u}\right)_i$$

is called a Poisson bracket between functions f_n and g_n.

We now derive the lattice hierarchy associated with the spectral problem (1.4). The stationary discrete zero curvature equation

$$(EV_n)U_n - U_n V_n = 0$$

with

$$V_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}$$

is equivalent to the following system

$$\begin{aligned}
&\lambda(a_{n+1} - a_n) + s_n b_{n+1} - r_n c_n = 0, \\
&-\lambda b_n + r_n(a_{n+1} + a_n) + \frac{q_n}{\lambda} b_{n+1} = 0, \\
&\lambda c_{n+1} - s_n(a_{n+1} + a_n) - \frac{q_n}{\lambda} c_n = 0, \\
&r_n c_{n+1} - s_n b_n - \frac{q_n}{\lambda}(a_{n+1} - a_n) = 0.
\end{aligned} \tag{2.1}$$

Substituting expansions

$$a_n = \sum_{j=0}^{\infty} a_n^{(j)} \lambda^{-2j}, \qquad b_n = \sum_{j=0}^{\infty} b_n^{(j)} \lambda^{-2j+1}, \qquad c_n = \sum_{j=0}^{\infty} c_n^{(j)} \lambda^{-2j+1}$$

into (2.1), we obtain the following recursive formulas

$$\begin{aligned}
&a_{n+1}^{(j)} - a_n^{(j)} + s_n b_{n+1}^{(j)} - r_n c_n^{(j)} = 0, \\
&-b_n^{(j+1)} + r_n(a_{n+1}^{(j)} + a_n^{(j)}) + q_n b_{n+1}^{(j)} = 0, \\
&c_{n+1}^{(j+1)} - s_n(a_{n+1}^{(j)} + a_n^{(j)}) - q_n c_n^{(j)} = 0, \\
&r_n c_{n+1}^{(j+1)} - s_n b_n^{(j+1)} - q_n(a_{n+1}^{(j)} - a_n^{(j)}) = 0, \quad j = 0, 1, \ldots, \\
&b_n^{(0)} = c_{n+1}^{(0)} = 0, \qquad a_{n+1}^{(0)} - a_n^{(0)} + s_n b_{n+1}^{(0)} - r_n c_n^{(0)} = 0.
\end{aligned} \tag{2.2}$$

If the initial values are chosen as

$$a_n^{(0)} = \frac{1}{2}, \qquad b_n^{(0)} = c_n^{(0)} = 0, \qquad a_n^{(j)}|_{q_n = r_n = s_n = 0} = 0, \quad j \geq 1,$$

then the system (2.2) can be solved successively through the following path:

$$\{(b_{n+1}^{(0)}, c_n^{(0)}) \to a_n^{(0)}\} \to \{(b_{n+1}^{(1)}, c_n^{(1)}) \to a_n^{(1)}\} \to \cdots \to \{(b_{n+1}^{(j)}, c_n^{(j)}) \to a_n^{(j)}\} \to \cdots.$$

In particular, the first two sets are

$$\begin{aligned} &b_n^{(1)} = r_n, \qquad c_{n+1}^{(1)} = s_n, \qquad a_n^{(1)} = -r_n s_{n-1}, \\ &b_n^{(2)} = -r_n(r_{n+1}s_n + r_n s_{n-1}) + q_n r_{n+1}, \\ &c_{n+1}^{(2)} = -s_n(r_{n+1}s_n + r_n s_{n-1}) + q_n s_{n-1}. \end{aligned}$$

Let

$$V_n^{(m)} = (\lambda^{2m} V_n)_+ = \sum_{j=0}^{m} \begin{pmatrix} a_n^{(j)} & b_n^{(j)}\lambda \\ c_n^{(j)}\lambda & -a_n^{(j)} \end{pmatrix} \lambda^{2m-2j}.$$

By using (2.2), a direct computation gives

$$(EV_n^{(m)})U_n - U_n V_n^{(m)} = \begin{pmatrix} 0 & b_n^{(m+1)} \\ -c_{n+1}^{(m+1)} & -\frac{q_n(a_{n+1}^{(m)} - a_n^{(m)})}{\lambda} \end{pmatrix}.$$

Checking the above expression, we find that $(EV_n^{(m)})U_n - U_n V_n^{(m)}$ is compatible with the matrix U_{n,t_m} in the spectral problem (1.4). As usual, we don't need to make an extra modification to the matrix $V_n^{(m)}$, and thus if we set

$$\psi_{n,t} = V_n^{(m)} \psi_n, \tag{2.3}$$

the compatibility condition between (1.4) and (2.3), i.e. the zero curvature equation

$$U_{n,t} - (EV_n^{(m)})U_n + U_n V_n^{(m)} = 0 \tag{2.4}$$

leads to the following hierarchy of discrete equations

$$\begin{aligned} r_{n,t_m} &= b_n^{(m+1)}, \\ s_{n,t_m} &= -c_{n+1}^{(m+1)}, \\ q_{n,t_m} &= -q_n(E-1)a_n^{(m)}, \quad m = 0, 1, \ldots. \end{aligned} \tag{2.5}$$

In this way, however, we see that the hierarchy (2.5) can not be reduced to our desired Ablowitz-Ladik hierarchy and Volterra hierarchy.

In order to derive the associated hierarchies from zero curvature equation (2.4), we embed a modification term to $V_n^{(m)}$ by

$$\tilde{V}_n^{(m)} = V_n^{(m)} + \begin{pmatrix} 0 & 0 \\ 0 & \delta_n^{(m)} \end{pmatrix},$$

where the freely adjustable function $\delta_n^{(m)}$ will play a crucial role in obtaining our desired reductions. A direct calculation gives

$$(E\tilde{V}_n^{(m)})U_n - U_n \tilde{V}_n^{(m)} = \begin{pmatrix} 0 & b_n^{(m+1)} - r_n\delta_n^{(m)} \\ -c_{n+1}^{(m+1)} + s_n\delta_n^{(m)} & \frac{q_n(E-1)(\delta_n^{(m)} - a_n^{(m)})}{\lambda} \end{pmatrix}.$$

Then the compatibility condition between (1.4) and the following auxiliary spectral problem

$$\psi_{n,t} = \tilde{V}_n^{(m)} \psi_n$$

leads to a new hierarchy

$$\begin{aligned} r_{n,t_m} &= b_n^{(m+1)} - r_n \delta_n^{(m)}, \\ s_{n,t_m} &= -c_{n+1}^{(m+1)} + s_n \delta_{n+1}^{(m)}, \\ q_{n,t_m} &= q_n (E-1)(\delta_n^{(m)} - a_n^{(m)}), \quad m = 0, 1, \dots, \end{aligned} \tag{2.6}$$

where $\delta_n^{(m)}$ should be chosen such that all the equations in (2.6) are compatible. Obviously, the hierarchy (2.6) is a generalization of the hierarchy (2.5).

In the following, we discuss some interesting reductions of the hierarchy (2.6). Especially the Ablowitz-Ladik hierarchy and Volterra hierarchy can be obtained from the hierarchy (2.6) as special cases.

Case 1. In the case when $q_n = 1$, $\delta_n^{(m)} = a_n^{(m)}$, the spectral problem (1.4) becomes the Ablowitz-Ladik spectral problem [2, 17]

$$E\psi_n = \begin{pmatrix} \lambda & r_n \\ s_n & \frac{1}{\lambda} \end{pmatrix} \psi_n.$$

The corresponding hierarchy (2.6) exactly turns to the Ablowitz-Ladik hierarchy

$$\begin{aligned} r_{n,t_m} &= b_n^{(m+1)} - r_n a_n^{(m)}, \\ s_{n,t_m} &= -c_{n+1}^{(m+1)} + s_n a_{n+1}^{(m)}, \quad m = 0, 1, \dots \end{aligned} \tag{2.7}$$

which gives the well-known Ablowitz-Ladik lattice for $m = 1$

$$r_{n,t} = r_{n+1}(1 - r_n s_n), \qquad s_{n,t} = s_{n-1}(r_n s_n - 1).$$

For the Hamiltonian structure of the Ablowitz-Ladik hierarchy (2.7), we may refer to the references [15, 17].

Case 2. In the case when $q_n = 0$, $s_n = -1$, $\delta_n^{(m)} = -c_n^{(m+1)}$, the spectral problem (1.4) reduces to the Volterra spectral problem [22]

$$E\psi_n = \begin{pmatrix} \lambda & r_n \\ -1 & 0 \end{pmatrix} \psi_n.$$

The corresponding hierarchy (2.6) just reduces to the Volterra hierarchy

$$r_{n,t_m} = b_n^{(m+1)} + r_n c_n^{(m+1)}, \quad m = 0, 1, \dots. \tag{2.8}$$

The second equation of this hierarchy is exactly the well-known Volterra lattice

$$r_{n,t} = r_n (r_{n+1} - r_{n-1}).$$

The Hamiltonian structure of the Volterra hierarchy (2.8) can be found in [11, 22].

Case 3. In the case when $q_n = 1 + r_n s_n$, $\delta_n^{(m)} = 0$, the spectral problem (1.4) reduces to a new spectral problem

$$E\psi_n = \begin{pmatrix} \lambda & r_n \\ s_n & \frac{1+r_n s_n}{\lambda} \end{pmatrix} \psi_n.$$

From the corresponding hierarchy (2.6), we get the following new hierarchy

$$\begin{aligned} r_{n,t_m} &= b_n^{(m+1)}, \\ s_{n,t_m} &= -c_{n+1}^{(m+1)}, \quad m = 0, 1, \ldots. \end{aligned} \tag{2.9}$$

The first two systems in this hierarchy are

$$\begin{aligned} &r_{n,t} = r_n, \qquad s_{n,t} = -s_n, \\ &r_{n,t} = -r_n^2 s_{n-1} + r_{n+1}, \qquad s_{n,t} = -s_n^2 r_{n+1} + s_{n-1}. \end{aligned}$$

For Case 3, we shall establish the multi-Hamiltonian formulism for the hierarchy (2.9).

3 Multi-Hamiltonian Structure for the Lattice Hierarchy (2.9)

In this section, we show that the hierarchy (2.9) possesses multi-Hamiltonian structure and is integrable in Liouville's sense. In order to apply trace identity, we introduce notations

$$u_n = (r_n, s_n)^T, \qquad G_n^{(m)} = (c_n^{(m)}, b_{n+1}^{(m)})^T.$$

Following the formula (2.2), we derive the following recursive relations in $G_n^{(m)}$

$$\begin{aligned} &J_n G_n^{(m+1)} = K_n G_n^{(m)}, \qquad G_n^{(1)} = (s_{n-1}, r_{n+1})^T, \\ &m = 1, 2, \ldots, \end{aligned} \tag{3.1}$$

where J_n and K_n are two matrix operators, given by

$$J_n = \begin{pmatrix} 0 & E^{-1} \\ -E & 0 \end{pmatrix}, \qquad K_n = \begin{pmatrix} r_n \Delta r_n & -r_n \Delta s_n + (1 + r_n s_n) \\ -s_n \Delta r_n - (1 + r_n s_n) & s_n \Delta s_n \end{pmatrix}.$$

By using the properties of operators E and Δ, it is easy to show that J_n and K_n are skew-symmetric operators.

Setting $L_n = J_n^{-1} K_n$, we can write the hierarchy (2.9) in the form

$$u_{n,t_m} = J_n G_n^{(m+1)} = J_n L_n G_n^{(m)} = \cdots = J_n L_n^m G_n^{(1)}. \tag{3.2}$$

Proposition 1 *All the operators* $J_n L_n^k$ $(k = 0, 1, 2, \ldots, m)$ *are skew-symmetric.*

Proof We prove this proposition by induction method. It is clear that both J_n and $J_n L_n = K_n$ are skew-symmetric operators for the cases when $k = 0$ and $k = 1$ in (3.2). Moreover, we have

$$J_n L_n = L_n^* J_n.$$

Suppose that $J_nL_n^{k-1}$ is skew-symmetric, then it holds that

$$\begin{aligned}(J_nL_n^k)^* &= (J_nL_n^{k-1}L_n)^* = L_n^*(J_nL_n^{k-1})^* = -L_n^*J_nL_n^{k-1} \\ &= -J_nL_nL_n^{k-1} = -J_nL_n^k,\end{aligned}$$

which implies that $J_nL_n^k$ is skew-symmetric. The proof is completed. □

We take $\mathrm{tr}(AB)$ as Killing-Cartan form and denote

$$\tilde{V}_n = V_nU_n^{-1} = \begin{pmatrix} \frac{1+r_ns_n}{\lambda}a_n - s_nb_n & -r_na_n + \lambda b_n \\ \frac{1+r_ns_n}{\lambda}c_n + s_na_n & -r_nc_n - \lambda a_n \end{pmatrix}.$$

Making use of the recursive (2.1), direct calculation leads to

$$\begin{aligned}\left\langle \tilde{V}_n, \frac{\partial U_n}{\partial \lambda}\right\rangle &= \frac{1+r_ns_n}{\lambda^2}(r_nc_n + 2\lambda a_n) - s_nb_n \\ &= \frac{r_nc_n + 2\lambda a_n}{\lambda^2} + \frac{s_n}{\lambda^2}(r_n^2c_n + 2r_na_n - \lambda^2 b_n) \\ &= \frac{r_nc_n + 2\lambda a_n}{\lambda^2} - \frac{s_nb_{n+1}}{\lambda^2},\end{aligned}$$

$$\begin{aligned}\left\langle \tilde{V}_n, \frac{\partial U_n}{\partial s_n}\right\rangle &= -2r_na_n + \lambda b_n - \frac{r_n^2}{\lambda}c_n = \frac{b_{n+1}}{\lambda}, \\ \left\langle \tilde{V}_n, \frac{\partial U_n}{\partial r_n}\right\rangle &= \frac{c_n}{\lambda}.\end{aligned}$$

Following the following trace identity [7]

$$\frac{\delta}{\delta u_n}\left\langle \tilde{V}_n, \frac{\partial U_n}{\partial \lambda}\right\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left(\lambda^{\gamma}\left(\left\langle \tilde{V}_n, \frac{\partial U_n}{\partial q_n}\right\rangle, \left\langle \tilde{V}_n, \frac{\partial U_n}{\partial r_n}\right\rangle\right)^T\right),$$

we have

$$\frac{\delta}{\delta u_n}\left(\frac{r_nc_n + 2\lambda a_n - s_nb_{n+1}}{\lambda^2}\right) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\left(\frac{c_n}{\lambda}, \frac{b_{n+1}}{\lambda}\right)^T\right],$$

which is equivalent to

$$\frac{\delta}{\delta u_n}\left(r_nc_n^{(m)} + 2a_n^{(m)} - s_nb_{n+1}^{(m)}\right) = (-2m+\gamma)G_n^{(m+1)}.$$

To fix the γ in the above equation, we set $m = 0$ and find $\gamma = 0$. Therefore we conclude that

$$G_n^{(m+1)} = \frac{\delta H_n^{(m)}}{\delta u_n}, \quad m = 0, 1, 2, \ldots \tag{3.3}$$

where

$$H_n^{(0)} = r_{n+1}s_n, \qquad H_n^{(m)} = \frac{1}{2m}(s_nb_{n+1}^{(m)} - r_nc_n^{(m)} - 2a_n^{(m)}).$$

Combining (3.2) and (3.3) gives the desired multi-Hamiltonian formulation of the hierarchy (2.9)

$$u_{n,t_m} = J_n \frac{\delta H_n^{(m)}}{\delta u_n} = J_n L_n \frac{\delta H_n^{(m-1)}}{\delta u_n} = J_n L_n^m \frac{\delta H_n^{(0)}}{\delta u_n}. \tag{3.4}$$

In the end, we discuss the integrability of the hierarchy (3.4). It is crucial to show the existence of infinite involutive conserved densities.

Theorem 1 (i) *The Hamiltonian functions* $\{H_n^{(m)}\}$ $(m = 0, 1, \ldots)$ *defined by* (3.3) *constitute common conserved densities for the whole hierarchy* (3.4). (ii) *The hierarchy* (3.4) *is an integrable Hamiltonian system in Liouville's sense.*

Proof By using (3.2)–(3.4), we find that

$$\begin{aligned}\{H_n^{(k)}, H_n^{(m)}\} &= \left(\frac{\delta H_n^{(k)}}{\delta u_n}, J_n \frac{\delta H_n^{(m)}}{\delta u_n}\right) = (L_n^k G_n^{(1)}, J_n L_n^m G_n^{(1)}) \\ &= (L_n^k G_n^{(1)}, L_n^* J_n L_n^{m-1} G_n^{(1)}) = (L_n^{k+1} G_n^{(1)}, J_n L_n^{m-1} G_n^{(1)}) \\ &= \{H_n^{(k+1)}, H_n^{(m-1)}\}.\end{aligned}$$

Repeating the above argument gives

$$\{H_n^{(m)}, H_n^{(k)}\} = \{H_n^{(k)}, H_n^{(m)}\} = \{H_n^{(m+k)}, H_n^{(0)}\}. \tag{3.5}$$

On the other hand, we obtain

$$\begin{aligned}\{H_n^{(m)}, H_n^{(k)}\} &= (L_n^m G_n^{(1)}, J_n L_n^k G_n^{(1)}) = (J_n^* L_n^m G_n^{(1)}, L_n^k G_n^{(1)}) \\ &= -\{H_n^{(k)}, H_n^{(m)}\}.\end{aligned} \tag{3.6}$$

Then combining (3.5) with (3.6) leads to

$$\{H_n^{(m)}, H_n^{(k)}\} = 0,$$

which implies that $\{H_n^{(m)}\}$ are in involution. Furthermore, we obtain

$$\left(\sum_{\in Z} H_n^{(m)}\right)_t = \left(\frac{\delta H_n^{(m)}}{\delta u_n}, u_{n,t}\right) = \left(\frac{\delta H_n^{(m)}}{\delta u_n}, J_n \frac{\delta H_n^{(k)}}{\delta u_n}\right) = \{H_n^{(m)}, H_n^{(k)}\} = 0,$$

which shows that $\{H_n^{(m)}\}$ are also conserved densities. The proof is completed. □

In this paper, we propose a lattice hierarchy with a free function which can be connected with some well-known lattice hierarchies by properly choosing the embedded function. This idea is also suitable for other more general spectral problems, which will be investigated in our future work.

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